

A topos for categories

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Introduction

- Motivated by certain problems in the theory of double categories
- (Strict) double categories are categories in **Cat**
- Many properties of **Doub** follow from the general theory of **Cat(C)**
- **Example:** **Cat(C)** has a 2-category structure
- **Example:** **C** cartesian closed with finite limits, then so is **Cat(C)**
- When proving such results, we secretly pretend that **C** is **Set**

Cat is not Set

Cat has many nice set-like properties

- Complete and cocomplete
- Locally finitely presentable
- Cartesian closed
- Sums are disjoint and universal

But

- Not a regular category

$$\begin{array}{ccccc} & 0 & & & \\ & \rightrightarrows & & & \\ \mathbf{1} & \xrightarrow{1} & \mathbf{2} & \xrightarrow{\quad} & \mathbb{N} \\ & \lleftarrow & & & \\ & 1 & & & \\ & \uparrow & \boxed{\text{Pb}} & \uparrow & \\ & \mathbf{1} + \mathbf{1} + \mathbf{1} + \mathbf{1} & \xrightarrow{\quad} & \mathbf{2} & \end{array}$$

- Not locally cartesian closed

Culprit: Quotients

Plan: Fix quotients and not break anything else

Fixing quotients

- Embed **Cat** in a suitable Grothendieck topos
- Introduced by Verity in his thesis to study double limits
- $\mathbf{Cat} \hookrightarrow \mathbf{sSet}$
- Other possibilities, e.g. $\mathbf{Set}^{\mathbf{fpCat}^{op}}$
- Smallest one is sheaves on the canonical topology on **Cat**

The topos hull of **Cat**

Theorem

- (1) $\langle F_i: \mathbf{C}_i \longrightarrow \mathbf{C} \rangle_{i \in I}$ is a covering family for the canonical topology on **Cat** if and only if it is onto on composable pairs.
- (2) The category of sheaves on **Cat** for the canonical topology is equivalent to

$$\mathbf{Set}^{\Delta_2^{op}},$$

2-truncated simplicial sets.

- (3) It is also equivalent to $M\text{-}\mathbf{Set}$ (right M -sets) where M is the monoid of endomorphisms of **3**

$\mathbf{Set}^{\Delta_2^{op}}$ is like **Set** but it's also like **Cat**!

Lazy categories

Definition

A *lazy category* is an object of $\mathbf{Set}^{\Delta_2^{op}}$. A *(lazy) functor* is a morphism in $\mathbf{Set}^{\Delta_2^{op}}$. The category of such is denoted **LCat**.

•

$$\mathbf{C} = \mathbf{C}_2 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\lambda} \\ \xrightarrow{\quad} \\ \xleftarrow{\rho} \\ \xrightarrow{\quad} \end{array} \mathbf{C}_1 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\text{id}} \\ \xrightarrow{\quad} \end{array} \mathbf{C}_0$$

- The elements of $\mathbf{C}_0$ are called *objects*, denoted $A, A', B, \dots$
- The elements of $\mathbf{C}_1$ are called *morphisms* (sometimes *idle* or *lazy morphisms*) and denoted $f: A \longrightarrow B, \dots$
- The elements of $\mathbf{C}_2$ are called *composers* and denoted

$$\begin{array}{ccc} & A_1 & \\ a_0 \nearrow & & \searrow a_1 \\ A_0 & \xrightarrow{a_2} & A_0 \\ & \alpha & \end{array}$$

- $\text{id}(A)$ is denoted $1_A: A \longrightarrow A$ and called the *identity morphism* on A

Lazy categories (Continued)

- $\lambda(f)$ is denoted by

$$\begin{array}{ccc} & B & \\ f \nearrow & & \searrow 1_B \\ A & \xrightarrow{f} & B \\ & \lambda_f & \end{array}$$

and called the *left identity composer*

- $\rho(f)$ is denoted by

$$\begin{array}{ccc} & A & \\ 1_A \nearrow & & \searrow f \\ A & \xrightarrow{f} & B \\ & \rho_f & \end{array}$$

and called the *right identity composer*

All of the equations that these satisfy are subsumed by the notation except for

$$\lambda_{1_A} = \rho_{1_A}$$

which we write as

$$\begin{array}{ccc} & A & \\ 1_A \nearrow & & \searrow 1_A \\ A & \xrightarrow{1_A} & A \\ & \iota_A & \end{array}$$

- A lazy functor $F: \mathbf{A} \longrightarrow \mathbf{X}$ is required to preserve all of the above.

Properties of **LCat**

Theorem

- (1) **Cat** is a reflective subcategory of **LCat**, i.e. there is a full embedding $\mathbf{Cat} \hookrightarrow \mathbf{LCat}$ which has a left adjoint L .
- (2) $\mathbf{Cat} \hookrightarrow \mathbf{LCat}$ creates all limits.
- (3) $\mathbf{Cat} \hookrightarrow \mathbf{LCat}$ creates coproducts and filtered colimits.
- (4) $\mathbf{Cat} \hookrightarrow \mathbf{LCat}$ preserves exponentiation, in fact **Cat** is an exponential ideal in **LCat**.
- (5) The reflector $L: \mathbf{LCat} \rightarrow \mathbf{Cat}$ preserves finite products.
- (6) The inclusion $\mathbf{Cat} \hookrightarrow \mathbf{LCat}$ preserves quotients by congruences.

Theorem

- (1) **LCat** is monadic over **Set**. It is algebras for the monad induced by the composable pairs functor $P: \mathbf{Cat} \rightarrow \mathbf{Set}$ and its left adjoint

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow{\quad} & \mathbf{LCat} \\ & \searrow P & \swarrow P \\ & \mathbf{Set} & \end{array}$$

- (2) Every lazy category **A** is a quotient of categories **X**, **Y**

$$\mathbf{Y} \rightrightarrows \mathbf{X} \twoheadrightarrow \mathbf{A}.$$

Some lazy category theory

The internal hom

- For $\mathbf{A}$ and $\mathbf{X}$ lazy categories, we have $\mathbf{X}^{\mathbf{A}}$
- Objects of $\mathbf{X}^{\mathbf{A}}$ are (lazy) functors $\mathbf{A} \longrightarrow \mathbf{X}$
- The morphisms, which we call *(lazy) natural transformations*, can be determined by analyzing

$$\frac{2 \longrightarrow \mathbf{X}^{\mathbf{A}}}{\mathbf{A} \times 2 \longrightarrow \mathbf{X}}$$

- The composers are determined by

$$\frac{3 \longrightarrow \mathbf{X}^{\mathbf{A}}}{\mathbf{A} \times 3 \longrightarrow \mathbf{X}}$$

Lazy natural transformations

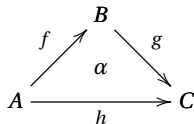
For $F, G: \mathbf{A} \longrightarrow \mathbf{X}$ lazy functors, a lazy natural transformation $t: F \longrightarrow G$ consists of:

- For every morphism $f: A \longrightarrow B$ in $\mathbf{A}$ a morphism

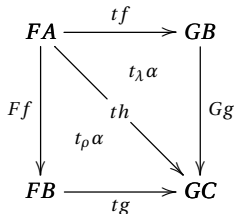
$$tf: FA \longrightarrow GB$$

in $\mathbf{X}$

- For every composer in $\mathbf{A}$



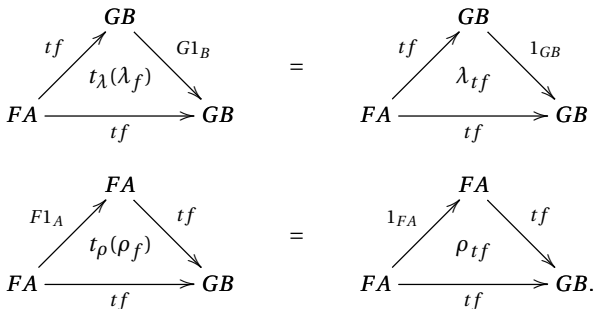
two composers in $\mathbf{X}$ (*left* and *right naturalizers*) $t_\lambda \alpha$ and $t_\rho \alpha$



subject to the conditions

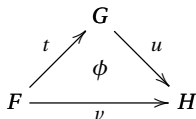
$$t_\lambda(\lambda_f) = \lambda_{tf} \text{ and } t_\rho(\rho_f) = \rho_{tf}$$

i.e.



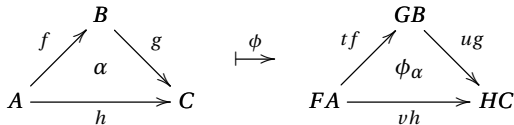
Natural composers

For lazy functors F, G, H and lazy natural transformations t, u, v as below, a natural composer ϕ



is given by:

For every composer α in $\mathbf{A}$ a composer $\phi(\alpha)$ in $\mathbf{X}$



subject to no extra conditions.

- When $\mathbf{A}$ and $\mathbf{X}$ are (actual) categories, lazy functors and natural transformations reduce to the usual thing, and composers are uniquely determined by vertical composition of natural transformations.

Horizontal composition

As **LCat** is cartesian closed it's enriched over itself giving composition morphism

$$\mathbf{LCat}(\mathbf{B}, \mathbf{C}) \times \mathbf{LCat}(\mathbf{A}, \mathbf{B}) \longrightarrow \mathbf{LCat}(\mathbf{A}, \mathbf{C})$$

- Associative and unitary.
- Composition of (lazy) functors is just function composition.
- Composition of (lazy) natural transformations is horizontal composition

$$\begin{array}{ccccc} \mathbf{A} & \xrightarrow{F} & & \xrightarrow{Q} & \mathbf{W} \\ & \Downarrow t & \mathbf{X} & \Downarrow q & \\ & \xrightarrow{G} & & \xrightarrow{R} & \end{array}$$

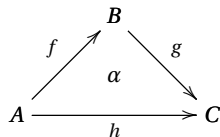
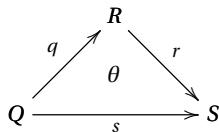
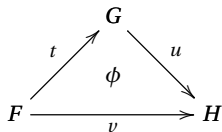
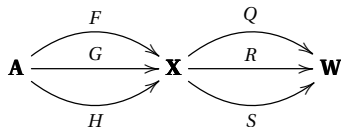
$$(qt)(f) = q(tf)$$

$$(A \xrightarrow{f} B) \quad \vdash \quad t \quad \xrightarrow{\quad} \quad (FA \xrightarrow{tf} GB) \quad \vdash \quad q \quad \xrightarrow{\quad} \quad (QFA \xrightarrow{qt f} RGB).$$

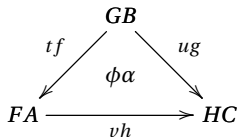
- Associative and unitary.
- Gives whiskering Qt and qF .

Interchange

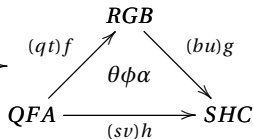
Composers also compose horizontally



$\vdash$



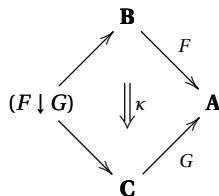
$\vdash$



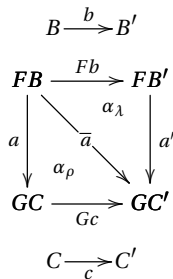
- This is lazy interchange.

Comma categories

For F and G as below we have $(F \downarrow G)$

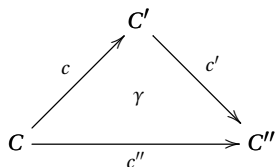
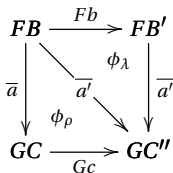
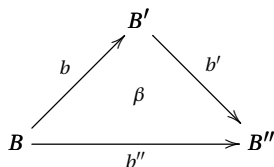
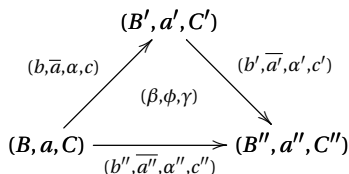


- Objects $(B, a: FB \longrightarrow GC, C)$
- Morphisms



Comma categories (Cont.)

– Composers

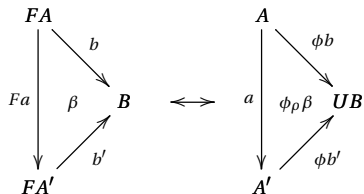
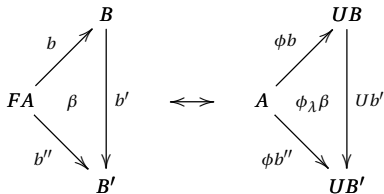


Lazy comma categories satisfy the usual universal property.

Lazy adjoints

- A **lazy adjunction** $F \dashv U: \mathbf{B} \longrightarrow \mathbf{A}$ is given by bijections $\phi, \phi_\lambda, \phi_\rho$

$$(FA \xrightarrow{b} B) \iff (A \xrightarrow{\phi b} UB)$$



satisfying

$$\phi_\lambda(\lambda_b) = \lambda_{\phi b} \quad \text{and} \quad \phi_\rho(\rho_b) = \rho_{\phi b}.$$

Example

$\Delta: \mathbf{A} \rightarrow \mathbf{A}^2$ is a left adjoint to $\partial_0: \mathbf{A}^2 \rightarrow \mathbf{A}$ if and only if $\mathbf{A}$ is a category.

Example

$\mathbf{A} \rightarrow \mathbf{1}$ has a right adjoint if and only if there is an object T such that for every A there is a unique morphism

$$!: A \rightarrow T$$

and for every morphism $f: A \rightarrow B$ there is a unique composer

$$\begin{array}{ccc} A & & \\ \downarrow f & \searrow ! & \\ B & \nearrow ! & T \end{array}$$

Theorem

- (1) *Adjoints compose*
- (2) *Adjoints are “lazily unique”*
- (3) *Adjoints are Lawvere adjoints*
- (4) *Adjoints are Street adjoints*

- F is a **Lawvere left adjoint** to U if there is an isomorphism over $\mathbf{A} \times \mathbf{B}$

$$\begin{array}{ccc}
 (F \downarrow \mathbf{B}) & \xrightarrow[\cong]{L} & (\mathbf{A} \downarrow U) \\
 & \searrow & \swarrow \\
 & \mathbf{A} \times \mathbf{B} &
 \end{array}$$

- F is a **Street left adjoint** to U if there are natural transformations $h: 1_{\mathbf{A}} \rightarrow UF$ and $e: FU \rightarrow 1_{\mathbf{B}}$ and two composites

$$\begin{array}{ccc}
 & UFU & \\
 hU \nearrow & & \searrow Ue \\
 U & \xrightarrow{1_U} & U \\
 & \eta &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & FUF & \\
 Fh \nearrow & & \searrow eF \\
 F & \xrightarrow{1_F} & F \\
 & \epsilon &
 \end{array}$$

Lazy profunctors

Definition

A *lazy profunctor* $P: \mathbf{A} \multimap \mathbf{B}$ consists of

1. For every A in $\mathbf{A}$ and B in $\mathbf{B}$ a set $P(A, B)$ of heteromorphisms $x: A \multimap B$
2. For every $a: A' \multimap A$ in $\mathbf{A}$ and B in $\mathbf{B}$ a set $P(a, B)$ of *right actors*

$$\begin{array}{ccc} & A & \\ & \nearrow x & \\ a \uparrow & \xi & B \\ & \nwarrow x' & \\ & A' & \end{array}$$

3. For every A in $\mathbf{A}$ and $b: B \multimap B$ in $\mathbf{B}$ a set $P(A, b)$ of *left actors*

$$\begin{array}{ccc} & B & \\ y \nearrow & \theta & \downarrow b \\ A & & B' \\ y' \searrow & & \end{array}$$

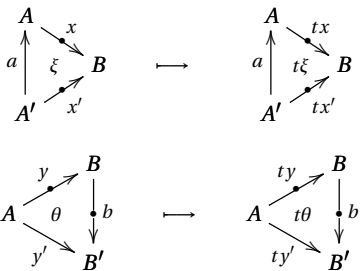
4. For every $x: A \multimap B$ in P left and right identity actors

$$\begin{array}{ccc} & A & \\ x \nearrow & \lambda_x & \downarrow 1_B \\ B & & B \\ x \searrow & & \end{array} \quad \text{and} \quad \begin{array}{ccc} A & & \\ \uparrow 1_A & \rho_x & \nearrow x \\ A & & B \\ & \nwarrow x & \end{array}$$

A morphism of lazy profunctors

$$t: P \multimap Q$$

$$(A \xrightarrow[p]{x} B) \mapsto (A \xrightarrow[Q]{tx} B)$$



preserving the left and right identity actors.

We get a category $\mathbf{Prof}(\mathbf{A}, \mathbf{B})$

Examples

For $F: \mathbf{A} \longrightarrow \mathbf{B}$ we get $F_*: \mathbf{A} \dashrightarrow \mathbf{B}$

$$F_*(A, B) = \mathbf{B}(FA, B)$$

and for $U: \mathbf{B} \longrightarrow \mathbf{A}$ we get $U^*: \mathbf{A} \dashrightarrow \mathbf{B}$

$$U^*(A, B) = \mathbf{A}(A, UB)$$

Theorem

F is left adjoint to U if and only if $F_ \cong U^*$ i.e.,*

$$\mathbf{B}(FA, B) \cong \mathbf{A}(A, UB)$$

Thank you!

